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On geodesic exponential maps of the Virasoro group

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Abstract

We study the geodesic exponential maps corresponding to Sobolev type right-invariant (weak) Riemannian metrics $\mu^{(k)}$ $(k \geq 0)$ on the Virasoro group `Vir` and show that for $k \geq 2$, but *not* for $k = 0, 1$, each of them defines a smooth Fréchet chart of the unital element $e \in \texttt{Vir}$. In particular, the geodesic exponential map corresponding to the KdV equation $(k = 0)$ is *not* a local diffeomorphism near the origin.

1 Introduction

The aim of this paper is to contribute towards a theory of Riemannian geometry for infinite dimensional Lie groups which has attracted a lot of attention since Arnold's seminal paper [1] on hydrodynamics. As a case study we consider the Virasoro group `Vir`, a central extension $\mathcal{D} \times \mathbb{R}$ of the Fréchet Lie group $\mathcal{D} \equiv \mathcal{D}(\mathbb{T})$ of orientation preserving C^∞-diffeomorphisms of the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and thus a Fréchet Lie group itself. Its Lie algebra `vir` can be identified with the Fréchet space $C^\infty(\mathbb{T}) \times \mathbb{R}$. The Virasoro group and its algebra come up in string theory ([25]) as well as in

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hydrodynamics, playing the rôle of a configuration space for the celebrated Korteweg - de Vries equation ([24, 16]). For $k \geq 0$ given, consider the scalar product $\langle \cdot, \cdot \rangle_k : \mathtt{vir} \times \mathtt{vir} \to \mathbb{R}$

$$\langle (u,a),(v,b) \rangle_k := \sum_{j=0}^{k} \int_0^1 \partial_x^j u \cdot \partial_x^j v \; dx + ab \; .$$

It induces a weak right-invariant Riemannian metric $\mu^{(k)}$ on $\mathtt{Vir}$. The notion of a *weak* metric, introduced in [10], means that the topology induced by $\mu^{(k)}$ on any tangent space $T_\Phi \mathtt{Vir}$, $\Phi \in \mathtt{Vir}$, is weaker than the Fréchet topology on $T_\Phi \mathtt{Vir}$. The aim of this paper is to show that results of classical Riemannian geometry concerning the geodesic exponential map induced by the metric $\mu^{(k)}$ continue to hold in $\mathtt{Vir}$ if $k \geq 2$. Note that it has been observed by Hamilton [12] and Milnor [21] that the Lie exponential map of the diffeomorphism group of the circle is *not* a local diffeomorphism near the origin. This fact can be used to show a similar result for the Virasoro group $\mathtt{Vir}$ – see Appendix C.

Theorem 1.1. *For any of the right-invariant metrics $\mu^{(k)}$, $k \geq 0$, there exists a neighborhood U_k of zero in $\mathtt{vir}$ such that for any initial vector $\xi \in U_k$ there exists a unique geodesic $\gamma(t;\xi)$ of $\mu^{(k)}$ with $\gamma|_{t=0} = e$ (e denotes the unital element in Vir) and $\dot\gamma|_{t=0} = \xi$, defined on the interval $t \in (-2,2)$ and depending C^1_F-smoothly[1] on the initial datum $\xi \in U_k$, i.e. $(-2,2) \times U_k \to$ $\mathtt{Vir}$, $(t,\xi) \mapsto \gamma(t;\xi)$ is C^1_F-smooth.*

Theorem 1.1 allows to define, for any given $k \geq 0$, on $U_k \subseteq \mathtt{vir}$ the geodesic exponential map

$$\exp_k : U_k \to \mathtt{Vir}, \quad \xi \mapsto \gamma(1;\xi) \; .$$

The following two theorems show that there is a fundamental dichotomy between the exponential maps $\exp_k$ for $k = 0,1$ and $k \geq 2$ – see Remark 3.3 for a precise explanation of this dichotomy.

Theorem 1.2. *For any $k \geq 2$ there exist a neighborhood U_k of zero in $\mathtt{vir}$ and a neighborhood V_k of the unital element e in Vir such that the geodesic exponential map $\exp_k|_{U_k} : U_k \to V_k$ is a C^1_F-diffeomorphism.*

[1]a map being C^k_F-smooth means that it is k-times continuously differentiable in the sense of Fréchet calculus – see Appendix A.

Theorem 1.3. *For $k = 0$ and $k = 1$ there is* no *neighborhood W_k of zero in* $\mathtt{vir}$ *so that the geodesic exponential map $\exp_k$ is a C^1_F-diffeomorphism from W_k onto a neighborhood of the unital element e in* $\mathtt{Vir}$.

Remark 1.4. *It has been shown in [24, 16] that the Euler equation, corresponding to the metric $\mu^{(0)}$, leads to the Korteweg - de Vries equation. Hence Theorem 1.3 reveals a sharp contrast between the Korteweg - de Vries equation and the Camassa - Holm equation which can be shown to be Euler equation for the metric $\mu^{(1)}$ on the Fréchet Lie group $\mathcal{D}$ of orientation preserving C^∞-diffeomorphisms of the torus $\mathbb{T}$ – see [16, 22]: It has been shown in [7] that the geodesic exponential map corresponding to the Sobolev type metric $\mu^{(k)}$ on $\mathcal{D}$ is a local diffeomorphism for $k \geq 1$ (but not for $k = 1$).*

Note that the natural inclusion of the diffeomorphism group $\mathcal{D}$ in $\mathtt{Vir}$, $\mathcal{D} \to \mathcal{D} \times \mathbb{R}$, $\phi \mapsto (\phi, 0)$ is *not* a subgroup of $\mathtt{Vir}$ and it turns out that the geodesic exponential map $\exp_k$ on $\mathtt{Vir}$ when restricted to $\mathcal{D}$ is different from the geodesic exponential map $\mathcal{D}$ constructed in [8]. Rephrasing Remark 1.4, Theorem 1.3 reveals a difference between the exponential maps on $\mathtt{Vir}$ and $\mathcal{D}$: According to Theorem 5 in [8], the geodesic exponential map $T_{\mathtt{id}}\mathcal{D} \to \mathcal{D}$ corresponding to the restriction of the metric $\mu^{(1)}$ to $\mathcal{D}$ is a local C^1_F-diffeomorphism near $0 \in T_{\mathtt{id}}\mathcal{D}$.

The paper is organized as follows: In Section 2 we fix notations and describe our set-up clarifying in this way the precise meaning of the statements in the above theorems. In Section 4, Theorem 1.1 and Theorem 1.3 are shown whereas Theorem 1.2 is proved in Section 3. For the convenience of the reader we have included at the end of the paper a section on the calculus in Fréchet spaces (Appendix A), on Euler equations on $\mathtt{vir}$ (Appendix B), and on the Lie exponential map (Appendix C).

2 Euler-Lagrange equations on the Virasoro group

Denote by $\mathcal{D} \equiv \mathcal{D}(\mathbb{T})$ the group of C^∞-smooth positively oriented diffeomorphisms of the 1-dimensional torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The topology on $\mathcal{D}$ is induced from the standard Fréchet topology on $C^\infty(\mathbb{T})$ corresponding to the countable system of $H^k(\mathbb{T})$ norms, $||u||_k^2 := \sum_{j=0}^k \int_0^1 (\partial_x^j u)^2 \; dx$, $k \geq 0$ (cf.

Appendix A). The Fréchet manifold $\mathcal{D}(\mathbb{T})$ is a Fréchet Lie group with multiplication $\circ : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ given by the composition of diffeomorphisms, i.e. if $(\phi, \psi) \in \mathcal{D} \times \mathcal{D}$ then $(\phi \circ \psi)(x) := \phi(\psi(x))$ (cf. [12]).[2]

Definition 2.1. *The* Virasoro group $\texttt{Vir}$ *is the Fréchet manifold* $\mathcal{D} \times \mathbb{R}$ *with multiplication* $\circ : \texttt{Vir} \times \texttt{Vir} \to \texttt{Vir}$ *given by the formula*

$$(\phi, \alpha) \circ (\psi, \beta) := \Big(\phi \circ \psi, \alpha + \beta - \frac{1}{2} \int_0^1 \log(\phi(\psi(x)))_x \, d \log \psi_x(x)\Big). \quad (2.1)$$

The map B, given by $B(\phi, \psi) := -\frac{1}{2} \int_0^1 \log(\phi \circ \psi)_x \; d \log \psi_x$ is sometimes refered to as the Bott cocycle.

Remark 2.2. *Passing to the universal cover* $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ *of the torus* $\mathbb{T} \equiv \mathbb{R}/\mathbb{Z}$ *we identify a diffeomorphisms* $\phi \in \mathcal{D}$ *with the set of its lifts* $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$, $\tilde{\phi} \in C^\infty(\mathbb{R}, \mathbb{R})$. *Two such lifts* $\tilde{\phi}_1$, $\tilde{\phi}_2$ *of* ϕ *are related by* $\tilde{\phi}_2(x) = \tilde{\phi}_1(x+k)+l$ *for some* $k, l \in \mathbb{Z}$. *It is readily seen that the expression*

$$-\frac{1}{2} \int_0^1 \log(\tilde{\phi}(\tilde{\psi}(x)))_x \, d \log \tilde{\psi}_x(x)$$

in formula (2.1) is independent of the choice of the lifts $\tilde{\phi}, \tilde{\psi} \in C^\infty(\mathbb{R}, \mathbb{R})$ *of* ϕ *and* ψ. *Often we will choose a lift of an element* ϕ *in* $\mathcal{D}$ *of the form* $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$, $\tilde{\phi}(x) = x + v(x)$ *with* $0 \leq v(0) < 1$ *and* v *a smooth* 1*-periodic function. In the sequel we will not distinguish between* ϕ *and its lifts to* $\mathbb{R}$.

One easily verifies that $\texttt{Vir}$ is a Fréchet Lie group whose algebra $\texttt{vir}$ can be identified with the Fréchet space $C^\infty(\mathbb{T}) \times \mathbb{R}$ with Lie bracket

$$[(u, a), (v, b)] = \Big(u_x v - v_x u, \int_0^1 u(x) v_{xxx}(x) \; dx\Big). \quad (2.2)$$

The map C, given by $C(u, v) := \int_0^1 u(x) v_{xxx}(x) \; dx$ is often refered to as Gelfand-Fuchs 2-cocycle. The unital element in $\texttt{Vir}$ is $e := (\texttt{id}, 0)$ where $\texttt{id}$ denotes the identity in $\mathcal{D}$.

Remark 2.3. *Usually the coefficient in front of the integral in (2.1) is taken to be equal to 1 instead of* $-\frac{1}{2}$ *(cf. [2, 16]). In this case, one has to insert a factor* -2 *in front of the integral in formula (2.2) for the Lie bracket.*

[2] Note that the composition on $\mathcal{D}$ is C_F^∞-smooth.

For a given $k \geq 0$ consider on $\mathtt{vir} = C^\infty(\mathbb{T}) \times \mathbb{R}$ the Sobolev type scalar product $\langle \cdot, \cdot \rangle_k : \mathtt{vir} \times \mathtt{vir} \to \mathbb{R}$

$$\langle (u,a),(v,b) \rangle_k := \sum_{j=0}^{k} \int_0^1 \partial_x^j u \cdot \partial_x^j v \, dx + ab, \quad \forall (u,a),(v,b) \in \mathtt{vir}. \tag{2.3}$$

This scalar product induces a right-invariant (weak) Riemannian metric[3] $\mu^{(k)}$ on $\mathtt{Vir}$. For any $\Phi \in \mathtt{Vir}$

$$\mu_\Phi^{(k)}(\xi, \eta) = \langle (d_e R_\Phi)^{-1} \xi, (d_e R_\Phi)^{-1} \eta \rangle_k, \quad \forall \xi, \eta \in T_\Phi \mathtt{Vir} \tag{2.4}$$

where $R_\Phi : \mathtt{Vir} \to \mathtt{Vir}$ denotes the right translation $\Psi \mapsto \Psi \circ \Phi$ in $\mathtt{Vir}$. It follows from its definition that $\mu^{(k)}$ is a C_F^∞-smooth[4] weak Riemannian metric on $\mathtt{Vir}$.

We define the *geodesics* with respect to a smooth (weak) Riemannian metric μ on Vir in the classical way as the *stationary points* of the *action functional* corresponding to μ.[5] The following definition makes sense on an arbitrary Fréchet manifold.

Definition 2.4. *A C_F^2-smooth curve $\gamma : [0,T] \to \mathtt{Vir}$, $T > 0$, is called a* geodesic *of the smooth (weak) Riemannian metric μ on $\mathtt{Vir}$ if for any C_F^2-smooth variation (with s denoting the variation parameter $-\epsilon < s < \epsilon$)*

$$\gamma : (-\epsilon, \epsilon) \times [0,T] \to \mathtt{Vir}, \quad (s,t) \mapsto \gamma(s,t) \ \ \textit{with} \ \ \gamma(0,t) = \gamma(t) \tag{2.5}$$

such that $\gamma(s,0) = \gamma(0)$ and $\gamma(s,T) = \gamma(T)$ for any $-\epsilon < s < \epsilon$ one has

$$\frac{d}{ds}\Big|_{s=0} E_\mu(\gamma(s,\cdot)) = 0. \tag{2.6}$$

Here E_μ denotes the *action functional*

$$E_\mu(\gamma(s,\cdot)) := \frac{1}{2} \int_0^1 \mu(\dot\gamma(s,t), \dot\gamma(s,t)) \, dt$$

[3]The word *weak* means that the topology induced by $\mu^{(k)}$ on any tangent space $T_\Phi \mathtt{Vir}$, $\Phi \in \mathtt{Vir}$, is weaker than the Fréchet topology on $T_\Phi \mathtt{Vir}$.

[4]The symbol C_F^k means that the corresponding map is k-times continuously differentiable in the sense of Fréchet calculus (see Appendix A and [12] for details). We reserve the symbol C^k for the standard notion of continuous differentiability in Banach spaces.

[5]Another approach is to prove that there exists a Riemannian (Levi-Civita) connection on $\mathtt{Vir}$ with respect to the metric $\mu^{(k)}$ and then to define the geodesics as the curves whose tangent vectors are parallel with respect to the connection (cf. [10, 8]).

and $\dot\gamma(s,t) := \frac{\partial\gamma}{\partial t}(s,t)$. The variational equation (2.6) leads (cf. Appendix B) to a partial differential equation for $\gamma(t)$, called *Euler-Lagrange equation.* Note that the existence and the uniqueness of geodesics on Fréchet manifolds might not hold. Indeed, the corresponding Euler-Lagrange equation can be viewed as a dynamical system (ODE) on the tangent bundle which is a Fréchet manifold as well. But on Fréchet manifolds, smooth ODE's may have no or more than one solution (cf. [12], p. 129).

It turns out that a C^2_F-smooth curve $t \mapsto \Phi(t) = (\phi(t), \alpha(t)) \in \mathtt{Vir}$ with

$$\Phi|_{t=0} = e \quad \text{and} \quad \frac{d\Phi}{dt}|_{t=0} = (u_0, a_0) \in \mathtt{vir}$$

is a geodesic with respect to the metric $\mu^{(k)}$ if and only if $\phi(t)$ and $\alpha(t)$ are solutions of the ordinary differential equations

$$\dot\phi(t) = u(t, \phi(t)) \tag{2.7}$$

$$\phi|_{t=0} = \mathtt{id} \tag{2.8}$$

and

$$\dot\alpha(t) = a(t) - \frac{1}{2}\int_0^1 u_x(t, \phi(t,x))\, d\log\phi_x(t,x) \tag{2.9}$$

$$\alpha(0) = 0 \tag{2.10}$$

where $(u(t), a(t)) \in \mathtt{vir}$ satisfies the so-called *Euler equation*

$$A_k u_t = -(2u_x A_k u + u A_k u_x) + a u_{xxx} \tag{2.11}$$

$$\dot a = 0 \tag{2.12}$$

with $A_k := \sum_{j=0}^k (-1)^j \partial_x^{2j}$ and initial data

$$u(0,x) = u_0(x) \quad \text{and} \quad a(0) = a_0.$$

We will derive the above system (2.7)-(2.12) in Appendix B. Let us point out that unlike in the case of the Lie group exponential map for $\mathtt{Vir}$ (see Appendix C) the element u in (2.7) generically depends on time.

Let $t \mapsto (\phi(t; u_0, a_0), \alpha(t; u_0, a_0)) \in \mathtt{Vir}$ be a C^2_F-smooth solution of (2.7)-(2.8) and (2.9)-(2.10) where $u(t,x) \equiv u(t,x;u_0,a_0)$ is a solution of the Euler equation (2.11)-(2.12) that we assume is defined on an open set in $\mathbb{R}$ containing the interval $[-1,1]$. Then the geodesic exponential map at (u_0, a_0) is defined by the formula

$$\exp_k : (u_0, a_0) \mapsto (\phi(t,x;u_0,a_0), \alpha(t;u_0,a_0))|_{t=1}. \tag{2.13}$$

Theorem 1.1 stated in the introduction and proved in Section 4 says that for any $k \geq 0$ the geodesic exponential map $\exp_k$ is well-defined in a small open neighborhood of zero in $\mathtt{vir}$.

3 Proof of Theorem 1.2

In this section we prove Theorem 1.2 stated in the Introduction.

As any geodesic flow of a right(left) invariant metric on a Lie group G admits a family of Nether integrals, the geodesic flow of the right invariant metric $\mu^{(k)}$ on $\mathtt{Vir}$ admits a family of integrals parametrized by the elements $\xi = (u, a) \in \mathtt{vir}$. To obtain such a family of integrals let us argue formally as follows: Given any geodesic $\gamma_0 : [0, T] \to \mathtt{Vir}$ and any element $\xi \in \mathtt{vir}$, consider the 1-parameter family of curves $\gamma : (-\epsilon, \epsilon) \times [0, T] \times \mathbb{R} \to \mathtt{Vir}$ $(\epsilon > 0)$

$$\gamma(s, t) := \gamma_0(t) \circ \eta(s)$$

where $\eta(s) := \exp_{Lie}^{\mathtt{Vir}}(s\xi)$ denotes the Lie group exponential map on $\mathtt{Vir}$ (cf. Appendix C). Note that $\dot{\gamma}(s, t) = \frac{d}{dt} R_{\eta(s)} \gamma_0(t) = d_{\gamma_0(t)} R_{\eta(s)} \dot{\gamma}_0(t)$. As $\mu^{(k)}$ is a right invariant metric it follows that

$$\begin{aligned} \mu^{(k)}_{\gamma(s,t)}(\dot{\gamma}(s, t), \dot{\gamma}(s, t)) &= \mu^{(k)}_{R_{\eta(s)}\gamma_0(t)}(d_{\gamma_0(t)} R_{\eta(s)} \dot{\gamma}_0(t), d_{\gamma_0(t)} R_{\eta(s)} \dot{\gamma}_0(t)) \\ &= \mu^{(k)}_{\gamma_0(t)}(\dot{\gamma}_0(t), \dot{\gamma}_0(t)) \, . \end{aligned}$$

Hence the action functional

$$E_{\mu^{(k)}}(\gamma(s, \cdot)) := \int_0^T \mu^{(k)}_{\gamma(s,t)}(\dot{\gamma}(s, t), \dot{\gamma}(s, t)) \, dt$$

is independent of s. In particular,

$$\frac{d}{ds}|_{s=0} E_{\mu^{(k)}}(\gamma(s, \cdot))$$

for any choice of $\xi = (u, a) \in \mathtt{vir}$. Computing the above variation explicitly as in Appendix B but with varying endpoints (i.e. without assuming (6.2)) one obtains the invariant function (3.1) defined below.

Lemma 3.1. *Let $\gamma(t) = (\phi(t), \alpha(t)) \in \mathtt{Vir}$ be a geodesic of the weak Riemannian metric $\mu^{(k)}$ on $\mathtt{Vir}$. Then $\gamma(t)$ the function*

$$I_k(\dot{\gamma}(t)) := \phi_x(t)^2 \cdot (A_k u(t)) \circ \phi(t) - a_0 S(\phi(t)) \tag{3.1}$$

is independent of t where $u = \phi_t \circ \phi^{-1}$ and where $S(\phi)$ is the Schwarzian derivative of ϕ, $S(\phi) := (\phi_x \phi_{xxx} - 3\phi_{xx}^2/2)/\phi_x^2$.

Proof of Lemma 3.1. We will show by a direct computation at the end of Appendix B that $I_k(\dot\gamma(t))$ is independent of t. $\square$

Proof of Theorem 1.2. We will prove Theorem 1.2 by applying, for any given $k \geq 2$, Proposition 5.5 (Appendix A) to the *Hilbert approximation*[6]

$$\mathtt{vir}_{2k+1} \supseteq \mathtt{vir}_{2k+2} \supseteq \ldots \supseteq \mathtt{vir}$$

of the Fréchet space $\mathtt{vir} = C^\infty(\mathbb{T}) \times \mathbb{R}$ (cf. Appendix A) where $\mathtt{vir}_l := H^l(\mathbb{T}) \times \mathbb{R}$.

Recall that $\mathcal{D}^s \equiv \mathcal{D}^s(\mathbb{T})$ ($s \geq 2$) denotes the Hilbert manifold

$$\mathcal{D}^s(\mathbb{T}) := \{\phi \in H^s(\mathbb{T}, \mathbb{T}) \mid \phi'(x) > 0 \ \ \forall x \in \mathbb{T}\}.$$

Representing an element $\phi \in \mathcal{D}^s$ in the form $\phi(x) = x + f(x)$ one can easily see that a neighborhood of the identity $\mathtt{id}$ in $\mathcal{D}^s$ can be identified with an open neighborhood of $0 \in H^s(\mathbb{T})$ (cf. Appendix A). The composition of mappings endows $\mathcal{D}^s$ with a topological Lie group structure.[7]

First we show the following result.

Proposition 3.2. *For any $k \geq 2$ given there exists a neighborhood U_{2k+1} of zero in $\mathtt{vir}_{2k+1}$ such that for any $l \geq 2k+1$ and any initial data $(u_0, a_0) \in U_l := U_{2k+1} \cap \mathtt{vir}_l$ there exists a (unique) solution $\Phi(t) = (\phi(t), \alpha(t)) \in C^1((-2,2), \mathcal{D}^l \times \mathbb{R})$ of (2.7)-(2.12) which depends C^1-smoothly on the initial data $(u_0, a_0) \in U_l$ in the sense that Φ belongs to $C^1((-2,2) \times U_l, \mathcal{D}^l \times \mathbb{R})$.*

Proof of Proposition 3.2. For a given $k \geq 2$ consider the pair of equations

$$\dot\phi = u(t, \phi(t)) \tag{3.2}$$
$$A_k u_t = -(2u_x A_k u + u A_k u_x) + a u_{xxx} \tag{3.3}$$

[6]For any $k \geq 2$ given, we choose $\mathtt{vir}_{2k+1}$ as the first approximation space of $\mathtt{vir}$ to insure that all our calculations will take place in $H^s(\mathbb{T})$ with $s \geq 1$. Note that $H^s(\mathbb{T})$ is a Banach algebra for $s \geq 1$.

[7]Unfortunately, the composition $\circ : \mathcal{D}^s \times \mathcal{D}^s \to \mathcal{D}^s$ and the inverse operation $(\cdot)^{-1} : \mathcal{D}^s \to \mathcal{D}^s$ are not C^∞. Nevertheless, it is easy to check that the composition $\circ : \mathcal{D}^{s+l} \times \mathcal{D}^{s+l} \to \mathcal{D}^s$ ($l \geq 0$) is a C^l-smooth map (see e.g. [10]).

depending on the real parameter $a = a_0$, with initial data $u|_{t=0} = u_0$ and $\phi|_{t=0} = \mathtt{id}$. Our first aim is to prove that for any $a \in \mathbb{R}$ and any $l \geq 2k+1$ there exist solutions $\phi \in C^1((-T,T), \mathcal{D}^l)$ and $u \in C^0((-T,T), H^l(\mathbb{T})) \cap C^1((-T,T), H^{l-1}(\mathbb{T}))$ of the system (3.2)-(3.3) which are defined for some $T > 0$ (possibly depending on the initial data $a \in \mathbb{R}$ and $u_0 \in H^l(\mathbb{T})$). To this end note that for any $l \geq 2k+1$ the system (3.2)-(3.3) can be reduced (by a change of variables) to a parameter depending ordinary differential equation (ODE) on the Hilbert manifold $\mathcal{D}^l \times H^l(\mathbb{T})$: For $u \in H^l(\mathbb{T})$ one has

$$A_k(uu_x) = uA_k u_x + Q_k(u)$$

where $Q_k(u)$ is a polynomial of the variables $u, u_x, ..., \partial_x^{2k} u$. As $H^{l-2k}(\mathbb{T})$ ($l \geq 2k+1$) is an algebra it follows that $Q_k \in C^\infty(H^l(\mathbb{T}), H^{l-2k}(\mathbb{T}))$. Using the latter identity, (3.3) can be rewritten in the form $A_k(u_t + uu_x) = -2u_x A_k u + Q_k(u) + au_{xxx}$ and

$$u_t + uu_x = A_k^{-1} \circ B_k(u; a)$$

where $B_k(u; a) := -2u_x A_k u + Q_k(u) + au_{xxx}$ is an element in $C^\infty(H^l(\mathbb{T}) \times \mathbb{R}, H^{l-2k}(\mathbb{T}))$ and $B_k(0; a) = 0$.

Remark 3.3. *The term au_{xxx} in the expression for $B_k(u; a)$ belongs to $H^{l-3}(\mathbb{T})$ which is included in $H^{l-2k}(\mathbb{T})$ when $k \geq 2$. If $k = 1$ the latter inclusion does not hold.*

Finally, the substitution $v(t) = u(t) \circ \phi(t)$ leads to the equation

$$\dot{\phi} = v \tag{3.4}$$
$$\dot{v} = F_k(\phi, v; a) \tag{3.5}$$

where $F_k(\phi, v; a) := (A_k^{-1} \circ B_k(v \circ \phi^{-1}; a)) \circ \phi$. The right hand side of (3.4)-(3.5) is well defined for any $(\phi, v) \in \mathcal{D}^l \times H^l(\mathbb{T})$ and belongs to the space $H^l(\mathbb{T}) \times H^l(\mathbb{T})$. In particular, (3.4)-(3.5) defines a *dynamical system* (ODE) on $\mathcal{D}^l \times H^l(\mathbb{T})$ which depends on the real parameter $a \in \mathbb{R}$. Let $R_\phi : \mathcal{D}^s \to \mathcal{D}^s$ be the right-translation in $\mathcal{D}^s$ by $\phi \in \mathcal{D}^s$ for $s \geq 1$. As

$$F_k(\phi, v; a) = R_\phi \circ A_k^{-1} \circ R_{\phi^{-1}} \circ R_\phi \circ B_k(a) \circ R_{\phi^{-1}} v,$$

the mapping

$$\mathcal{F}_k : (\phi, v; a) \mapsto (\phi, F_k(\phi, v; a))$$

can be decomposed, $\mathcal{F}_k = \mathcal{A}_k \circ \mathcal{B}_k(a)$, where

$$\mathcal{A}_k : (\phi, v) \mapsto (\phi, R_\phi \circ A_k^{-1} \circ R_{\phi^{-1}} v)$$

and

$$\mathcal{B}_k(a) : (\phi, v) \mapsto (\phi, R_\phi \circ B_k(a) \circ R_{\phi^{-1}} v).$$

As both, A_k^{-1} and $B_k(a)$ are conjugated by R_ϕ one can show by a straightforward computation that

$$\mathcal{A}_k \in C^1(\mathcal{D}^l \times H^{l-2k}(\mathbb{T}), \mathcal{D}^l \times H^l(\mathbb{T}))$$

and

$$\mathcal{B}_k(a) \in C^1(\mathcal{D}^l \times H^l(\mathbb{T}) \times \mathbb{R}, \mathcal{D}^l \times H^{l-2k}(\mathbb{T})).$$

In particular one sees that

$$\mathcal{F}_k \in C^1(\mathcal{D}^l \times H^l(\mathbb{T}) \times \mathbb{R}, \mathcal{D}^l \times H^l(\mathbb{T})).$$

To indicate how this is done let us verify that $\mathcal{A}_k : \mathcal{D}^l \times H^{l-2k}(\mathbb{T}) \to \mathcal{D}^l \times H^l(\mathbb{T})$ possesses directional derivatives w. r. to ϕ. Let $(\phi, v) \in \mathcal{D}^l \times H^{l-2k}(\mathbb{T})$ and consider the variation $\phi + \epsilon\, \delta\phi$ with $\delta\phi \in H^l(\mathbb{T})$. Use that

$$\frac{d}{d\epsilon}|_{\epsilon=0} u \circ (\phi + \epsilon\, \delta\phi) = (R_\phi u_x) \cdot \delta\phi$$

$$\frac{d}{d\epsilon}|_{\epsilon=0} (\phi + \epsilon\, \delta\phi)^{-1} = -\frac{1}{\phi_x \circ \phi^{-1}} \cdot \delta\phi \circ \phi^{-1}$$

and

$$\frac{d}{dx}(v \circ \phi^{-1}) = v_x \circ \phi^{-1} \cdot \frac{1}{\phi_x \circ \phi^{-1}}$$

to obtain

$$\begin{aligned}
&\tfrac{d}{d\epsilon}|_{\epsilon=0} \mathcal{A}_k(\phi + \epsilon\, \delta\phi, v) = \\
&= \tfrac{d}{d\epsilon}|_{\epsilon=0} \Big(R_{\phi+\epsilon\,\delta\phi} A_k^{-1} (v \circ \phi^{-1}) \Big) + R_\phi A_k^{-1} \tfrac{d}{d\epsilon}|_{\epsilon=0} \Big(R_{(\phi+\epsilon\,\delta\phi)^{-1}} v \Big) \\
&= R_\phi \Big(\tfrac{d}{dx} A_k^{-1} (v \circ \phi^{-1}) \Big) \cdot \delta\phi - R_\phi A_k^{-1} \Big(v_x \circ \phi^{-1} \cdot \tfrac{1}{\phi_x \circ \phi^{-1}} \cdot \delta\phi \circ \phi^{-1} \Big) \\
&= R_\phi \Big(A_k^{-1} (v \circ \phi^{-1})_x \cdot \delta\phi \circ \phi^{-1} \Big) - R_\phi A_k^{-1} \Big((v \circ \phi^{-1})_x \cdot \delta\phi \circ \phi^{-1} \Big).
\end{aligned}$$

Hence we get

$$\begin{aligned}
\frac{d}{d\epsilon}|_{\epsilon=0} \mathcal{A}_k(\phi + \epsilon\, \delta\phi, v) \;&=\; R_\phi \circ A_k^{-1} \circ \Big(A_k \Big(\Big(A_k^{-1} (v \circ \phi^{-1})_x \Big) \cdot (\delta\phi \circ \phi^{-1}) \Big) \\
&-\; (v \circ \phi^{-1})_x \cdot (\delta\phi \circ \phi^{-1}) \Big).
\end{aligned}$$

Clearly

$$A_k\left(\left(A_k^{-1}(v\circ\phi^{-1})_x\right)\cdot(\delta\phi\circ\phi^{-1})\right)=(v\circ\phi^{-1})_x\cdot(\delta\phi\circ\phi^{-1})+\texttt{l.o.}\tag{3.7}$$

where $\texttt{l.o.}$ stands for terms containing derivatives of ϕ and $\phi_t(=v)$ in x of order up to $2k$. This shows that $\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{A}_k(\phi+\epsilon\,\delta\phi,v)\in H^l(\mathbb{T})$ for any $\delta\phi\in H^l(\mathbb{T})$ as claimed.

By the existence and uniqueness theorem for ODE's in Banach spaces there exists a neighborhood W_l of $(\mathtt{id},0)$ in $\mathcal{D}^l\times H^l(\mathbb{T})$ and $\epsilon_l>0$ such that for any initial data $(\phi_0,v_0)\in W_l$ and any $a\in(-\epsilon_l,\epsilon_l)$ the ordinary differential equation (3.4)-(3.5) has a unique solution $\Psi(t)=(\phi(t),v(t))$ defined for $t\in(-2,2)$.[8] Moreover, the solution $\Psi(t)$ depends C^1 smoothly on the parameter $a\in(-\epsilon_l,\epsilon_l)$ and the initial data $(\phi_0,v_0)\in W_l$.

Consider the neighborhood W_{2k+1}. Define for $l\geq 2k+1$ the neighborhood $V_l:=W_{2k+1}\cap(\mathcal{D}^l\times H^l(\mathbb{T}))$ of the point $(\mathtt{id},0)$ in $\mathcal{D}^l\times H^l(\mathbb{T})$ and set $\epsilon:=\epsilon_{2k+1}$. We will prove that for any initial data $(\phi_0,v_0)\in V_l$ and any $a\in(-\epsilon,\epsilon)$ there exists a solution of (3.4)-(3.5) in $\mathcal{D}^l\times H^l(\mathbb{T})$ that is defined for $t\in(-2,2)$ and depends C^1 smoothly on the parameter $a\in(-\epsilon,\epsilon)$ and the initial data $(\phi_0,v_0)\in V_l$. To this end we use induction in $l\geq 2k+1$: For $l=2k+1$ the statement is already proved. Denote by $\Psi(t)=(\phi(t),v(t))$ the corresponding solution. Now, take $a\in(-\epsilon,\epsilon)$ and $(\phi_0,v_0)\in V_{2k+2}$. As the left hand side of (3.4)-(3.5) is a C^1-smooth vector field on $\mathcal{D}^{2k+2}\times H^{2k+2}(\mathbb{T})$ that depends C^1 smoothly on the real parameter a, there exists a unique solution $\tilde\Psi(t)=\tilde\Psi(t;\phi_0,v_0,a)$ of (3.4)-(3.5) in $\mathcal{D}^{2k+2}\times H^{2k+2}(\mathbb{T})$ defined on some maximal interval of existence $t\in(T_1,T_2)$ with $T_1<0<T_2$. We claim that $T_2\geq 2$ and $T_1\leq -2$. As the two statements are proved similarly we concentrate on T_2 only. Arguing by contradiction suppose that $T_2<2$. Considered as a curve in $\mathcal{D}^{2k+1}\times H^{2k+1}(\mathbb{T})$, the solution $\tilde\Psi(t)=(\tilde\phi(t),\tilde v(t))$ solves (3.4)-(3.5) in $\mathcal{D}^{2k+1}\times H^{2k+1}(\mathbb{T})$ and therefore coincides with the solution $\Psi:[-2,2]\to\mathcal{D}^{2k+1}\times H^{2k+1}(\mathbb{T})$ of (3.4)-(3.5) in $\mathcal{D}^{2k+1}\times H^{2k+1}(\mathbb{T})$ on the interval $t\in(\max\{-2,T_1\},T_2)$. A direct calculation shows that the quantity

$$I_k(t)\equiv I_k(t,\phi,u):=\phi_x(t)^2\cdot(A_k u(t))\circ\phi(t)-aS(\phi(t))\tag{3.8}$$

is independent of $t\in(-2,2)$.[9] As $u=\phi_t\circ\phi^{-1}$ one has $u_x=(\phi_{tx}\circ\phi^{-1})\cdot(\phi^{-1})_x$. Using that $(\phi^{-1})_x=1/(\phi_x\circ\phi^{-1})$ and hence $(\partial_x^{2k}\phi^{-1})\circ\phi=-\partial_x^{2k}\phi/\phi_x^{2k+1}+\texttt{l.o.}$

[8]Note that the solution $t\mapsto(\mathtt{id},0)$ of (3.4)-(3.5) with initial data $\phi_0=\mathtt{id}$ and $v_0=0$ exists globally in time.

[9]By the same arguments as in the proof of Lemma 6.2 one can show that the variant of Lemma 3.1 holds in $\mathcal{D}^l$ $(l\geq 2k+1)$.

one gets

$$(\partial_x^{2k} u) \circ \phi = (\partial_x^{2k} \phi_t)/\phi_x^{2k} - \phi_{tx} \cdot \partial_x^{2k} \phi / \phi_x^{2k+1} + \mathtt{l.o.}$$

where l.o. stands for terms containing derivatives of ϕ or $\phi_t (= v)$ in x of order up to $2k-1$. Together with (3.8) and the expression for A_k one then obtains

$$(-1)^k \phi_x^{2k-1} I_k(t) = \phi_x \cdot \partial_x^{2k} \phi_t - \phi_{tx} \cdot \partial_x^{2k} \phi + \mathtt{l.o.}$$

Hence we have for any $t \in (-2, 2)$

$$\phi_x(t) \cdot \partial_x^{2k} \phi_t(t) - \phi_{tx}(t) \cdot \partial_x^{2k} \phi(t) = (-1)^k \phi_x^2(t) \Big(\phi_x^{2k-3}(t) I_k(t) + J_k(\phi(t), v(t); a) \Big)$$

where $J_k(\phi, v; a) = P_k(\phi, v; a)/\phi_x^2$ and P_k is a polynomial of the variables $\phi, \partial_x \phi, ..., \partial_x^{2k-1} \phi$, $v, \partial_x v, ..., \partial_x^{2k-1} v$ and a. As $I_k(t) = I_k(0)$ we obtain that

$$\Big(\frac{\partial_x^{2k} \phi(t)}{\phi_x(t)} \Big)_t = (-1)^k (\phi_x^{2k-3}(t) I_k(0) + J_k(\phi(t), v(t); a))$$

for any $t \in (-2, 2)$

$$\frac{\partial_x^{2k} \phi(t)}{\phi_x(t)} = \frac{\partial_x^{2k} \phi_0}{\phi_{0x}} + (-1)^k \int_0^t (\phi_x^{2k-3}(s) I_k(0) + J_k(\phi(s), v(s); a)) \, ds \, . \quad (3.9)$$

For $t \in (T_1, T_2)$ equality (3.9) implies

$$\partial_x^{2k} \tilde{\phi}(t) = (-1)^k \phi_x(t) \int_0^t (\phi_x^{2k-3}(s) I_k(0) + J_k(\phi(s), v(s); a)) \, ds \, . \quad (3.10)$$

As $(\phi_0, v_0) \in V_{2k+2}$ we get from (3.8) that $I_k(0) \in H^2(\mathbb{T})$. Then equality (3.10) implies that $\partial_x^{2k} \tilde{\phi}(t) \in H^2(\mathbb{T})$. Moreover, as $T_2 < 2$ by assumption one gets from (3.10) that the limit

$$\lim_{t \to T_2 - 0} \tilde{\phi}(t)$$

exists in $H^{2k+2}(\mathbb{T})$. As this limit equals $\phi(T_2)$ one concludes that $\phi(T_2)$ is an element in $\mathcal{D}^{2k+2}$. As $\tilde{v} = \tilde{\phi}_t$ and $\tilde{v}_t = F_k(\tilde{\phi}, \tilde{v}; a)$ evolve both in $H^{2k+2}(\mathbb{T})$ for $T_1 < t < T_2$ and as $T_2 < 2$ by assumption one concludes, by taking the t-derivatives of both sides of the identity (3.10) that the limit $\lim_{t \to T_2 - 0} \tilde{v}(t)$ exists in $H^{2k+2}(\mathbb{T})$ and equals $v(T_2)$. In particular, $v(T_2) \in H^{2k+2}(\mathbb{T})$. This contradicts the fact that (T_1, T_2) is the maximal interval of existence of the solution $\tilde{\Psi}(t)$. Hence, $T_2 \geq 2$.

The same arguments permit us to complete the induction and to show that for any $l \geq 2k+1$ and for any $(\phi_0, v_0) \in U_l$ and $a \in (-\epsilon, \epsilon)$ there exists a solution of (3.4)-(3.5) in $\mathcal{D}^l \times H^l(\mathbb{T})$ that is defined for $t \in (-2, 2)$ and depends C^1 smoothly on the parameter $a \in (-\epsilon, \epsilon)$ and the initial data $(\phi_0, v_0) \in V_l$. Combining this with the formula (2.9) for $\dot{\alpha}(t)$ one gets

$$\alpha(t) = a \cdot t - \frac{1}{2} \int_0^t \int_0^1 v_x(\tau) \frac{\phi_{xx}(\tau)}{\phi_x(\tau)^2} \, dx \, d\tau. \tag{3.11}$$

This completes the proof of Proposition 3.2. $\square$

It follows from Proposition 3.2 that for any given $k \geq 2$ one can define the mapping

$$\mathtt{E}_k : U_{2k+1} \to \mathcal{D}^{2k+1}(\mathbb{T}) \times \mathbb{R}, \;\; \mathtt{E}_k(u_0, a_0) := \Phi(t)|_{t=1}$$

where $U_{2k+1} \subseteq \mathtt{vir}_{2k+1}$[10] and $\Phi(t) = (\phi(t), \alpha(t))$ is the solution of (2.7)-(2.12). According to Proposition 3.2 for any $l \geq 2k+1$ the restriction

$$\mathtt{E}_k|_{U_l} : U_l \to H^l(\mathbb{T}) \times \mathbb{R}$$

of $\mathtt{E}_k$ to $U_l := U_{2k+1} \cap \mathtt{vir}_l$ is well-defined and C^1-smooth.

The following lemmas follow directly from (3.9).

Lemma 3.4. *If $(u_0, a_0) \in U_{2k+1}$ and $l \geq 2k+1$ then*

$$\mathtt{E}_k(u_0, a_0) \in H^l(\mathbb{T}) \times \mathbb{R} \quad \textit{iff} \quad u_0 \in H^l(\mathbb{T}) \; .$$

Proof of Lemma 3.4. First we prove the "if" part of the Lemma. By the definition of U_{2k+1}, the statement is true for $l = 2k+1$. Assume that it is true for any $2k+1 \leq s < l$ and let $\mathtt{E}_k(u_0, a_0) := \Phi(1) \in H^l(\mathbb{T}) \times \mathbb{R}$. Then $\phi(1) \in H^l(\mathbb{T})$ and according to (3.9) one has

$$I_k(0, u_0, a_0) \int_0^1 \phi_x^{2k-3}(s) \, ds = \frac{(-1)^k}{\phi_x(1)} \Big(\partial_x^{2k} \phi(1) - \int_0^1 J_k(\phi(s), v(s); a) \, ds \Big) \, . \tag{3.12}$$

[10]If necessary, we shrink the neighborhood U_{2k+1} so that the projection of the image $\mathtt{E}_k(U_{2k+1})$ on $\mathcal{D}^{2k+1}$ lies in the chart $\{f \in H^{2k+1}(\mathbb{T}) \mid -1/2 < f(0) < 1/2, f' > -1\}$ of the Hilbert manifold $\mathcal{D}^{2k+1}$(see Appendix A).

By assumption

$$\frac{\partial_x^{2k}\phi(1)}{\phi_x(1)} \in H^{l-2k}(\mathbb{T}), \quad \int_0^1 \phi_x^{2k-3}(s)\, ds \in H^{l-2}(\mathbb{T}),$$

and

$$\frac{1}{\phi_x(1)} \int_0^1 J_k(\phi(s), v(s); a)\, ds \in H^{l-2k}(\mathbb{T})$$

one gets from (3.12) that $A_k u_0 = I_k(0, u_0, a_0) \in H^{l-2k}(\mathbb{T})$. Hence, $u_0 \in H^l(\mathbb{T})$. The "only if" statement of the lemma follows from Proposition 3.2. $\square$

Lemma 3.5. *For any given* $(u_0, a_0) \in U_l$, $l \geq 2k+1$,

$$(d_{(u_0,a_0)}\mathtt{E}_k)(u, a) \in \mathtt{vir}_l \setminus \mathtt{vir}_{l+1}$$

for any $(u, a) \in \mathtt{vir}_l \setminus \mathtt{vir}_{l+1}$.

Proof of Lemma 3.5. The lemma is proved by passing to the variations of $\phi(t)$ in (3.9) and then arguing as in the proof of the previous lemma. $\square$

To finish the proof of Theorem 1.2 note that conditions (a), (b), and (c) of Proposition 5.5 in Appendix A hold in view of Lemma 3.4, Proposition 3.2, and Lemma 3.5 respectively. Hence Proposition 5.5 can be applied and Theorem 1.2 is proved. $\square$

4 Exponential maps corresponding to KdV and CH

If $k = 0$ the Euler equation (2.11)-(2.12) is the *Korteweg-de Vries equation* (KdV) with parameter $a_0 \in \mathbb{R}$,

$$u_t + 3uu_x - a_0 u_{xxx} = 0 \tag{4.1}$$

$$u|_{t=0} = u_0 \tag{4.2}$$

and if $k = 1$ we get the following variant of the *Camassa-Holm equation* (CH)

$$(1 - \partial_x^2)u_t = -2u_x(1 - \partial_x^2)u - u(1 - \partial_x^2)u_x + a_0 u_{xxx} \tag{4.3}$$

$$u|_{t=0} = u_0 \tag{4.4}$$

with a_0 being again a real parameter. It is well known that both equations can be viewed as integrable Hamiltonian systems and both are bi-Hamiltonian ([11, 5, 9, 14]).

Denote by $\mathtt{vir}_l$ the space $H^l(\mathbb{T}) \times \mathbb{R}$.

Lemma 4.1. *There exist a neighborhood U_3 of the zero in $\mathtt{vir}_3$ and a time interval $(-T, T)$, $T > 0$, such that for any $l \geq 3$ and any initial data $(u_0, a_0) \in U_l := U_3 \cap \mathtt{vir}_l$ the parametrized CH equation (4.3)-(4.4) has a unique solution $u \in C^0((-T,T), H^l(\mathbb{T})) \cap C^1((-T,T), H^{l-1}(\mathbb{T}))$ which depends C^1-smoothly on the initial data $(u_0, a_0) \in U_l$ in the sense that $u \in C^1((-T,T) \times U_l, H^{l-1}(\mathbb{T}))$.*

Proof of Lemma 4.1. With the substitution

$$u(t,x) := v(t, x - 3a_0t/2) + a_0/2 \tag{4.5}$$

equation (4.3)-(4.4) transforms into the standard form of the Camassa-Holm shallow water equation (cf. [5, 6])

$$\begin{aligned} (1-\partial_x^2)v_t &= -2v_x(1-\partial_x^2)v - v(1-\partial_x^2)v_x && (4.6)\\ v|_{t=0} &= u_0 - a_0/2 =: v_0 && (4.7) \end{aligned}$$

Now, the statement of the lemma follows from the arguments used to prove Theorem 1.2. Indeed, according to [22] (see also [18]) the nonlinear equation (4.6)-(4.7) is the Euler equation part of the geodesic equations corresponding to the right-invariant metric $\nu^{(1)}$ on the diffeomorphism group $\mathcal{D}(\mathbb{T})$ generated by the scalar product on $T_{\mathtt{id}}\mathcal{D} \cong C^\infty(\mathbb{T})$

$$\langle u, v\rangle_1 := \int_0^1 (uv + u_xv_x)\, dx, \quad u, v \in C^\infty(\mathbb{T}).$$

Using the same arguments as in the proof of Proposition 3.2 one shows that the geodesic equation on $T\mathcal{D}$ can be also considered as an ODE

$$\begin{aligned} (\dot{\psi}, \dot{w}) &= (w, G(\psi, w)) && (4.8)\\ (\psi, w)|_{t=0} &= (\psi_0, v_0) && (4.9) \end{aligned}$$

on the tangent bundle $T\mathcal{D}^l$ of the Hilbert manifold $\mathcal{D}^l$ ($l \geq 3$) where the vector field $(\psi, w) \mapsto (w, G(\psi, w))$ is C^1-smooth in a neighborhood of $(\mathtt{id}, 0) \in T\mathcal{D}^l$ (cf. [7]). The local existence, uniqueness and dependence on parameters theorems for ODE in Banach spaces [19, Chapter IV] then imply that there exist $T = T_l > 0$ and a neighborhood W_l of $(\mathtt{id}, 0)$ in $T\mathcal{D}^l$ such that for any $(\psi_0, v_0) \in W_l$ the nonlinear equation (4.8)-(4.9) has a unique solution $(\psi(t; \psi_0, v_0), w(t; \psi_0, v_0)) \in T\mathcal{D}^l$ for $t \in (-T, T)$ with $w(\cdot; \cdot, \cdot) \in C^1((-T, T) \times W_l, H^l(\mathbb{T}))$ and $\psi(\cdot; \cdot, \cdot) \in C^1((-T, T) \times W_l, \mathcal{D}^l)$. Then

$$v(t; v_0) = w(t; \mathtt{id}, v_0) \circ \psi(t; \mathtt{id}, v_0)^{-1}, \tag{4.10}$$

is the unique solution of (4.6)-(4.7) and has the property

$$v(\cdot; v_0) \in C^0((-T, T), H^l(\mathbb{T})) \cap C^1((-T, T), H^{l-1}(\mathbb{T})).$$

As the maps

$$\mathcal{D}^l \to \mathcal{D}^{l-1}, \quad \psi \mapsto \psi^{-1}$$

and

$$H^l(\mathbb{T}) \times \mathcal{D}^l \to H^{l-1}(\mathbb{T}), \quad (w, \psi) \mapsto w \circ \psi$$

are C^1-smooth we conclude from (4.10) that

$$v \in C^1((-T, T) \times V_l, H^{l-1}(\mathbb{T}))$$

where $V_l = W_l \cap T_{\mathtt{id}}\mathcal{D}$. Then arguing as in the proof of Proposition 3.2 one proves that V_l can be taken of the form

$$V_l = V_3 \cap H^l(\mathbb{T})$$

and T_l can be chosen to be T_3 and hence is independent of $l \geq 3$. Using these properties of v the properties of u stated in Lemma 4.1 follow from formula (4.5). $\square$

The following remark will be of use in the proof of Theorem 1.3.

Remark 4.2. *As the vector field $(\psi, w) \mapsto (w, G_k(\psi, w))$ is of class C^1 in a neighborhood of $(\mathtt{id}, 0) \in T\mathcal{D}^k$, the* local smoothness theorem *in [19, Chapter IV] implies that the partial derivatives $D_1 D_3 w(t; \psi_0, v_0)$ and $D_1 D_3 \psi(t; \psi_0, v_0)$ of $w(t; \psi_0, v_0)$ and $\psi(t; \psi_0, v_0)$ exist and, from the variational equation satisfied by $D_3 w(t; \psi_0, v_0)$,*

$$D_1 D_3 w(t; \psi_0, v_0) = D_3 D_1 w(t; \psi_0, v_0)$$

and

$$D_1 D_3 \psi(t; \psi_0, v_0) = D_3 D_1 \psi(t; \psi_0, v_0).^{11}$$

In particular, the same is true for the solution

$$v(t; v_0) = w(t; \mathtt{id}, v_0) \circ \psi(t; \mathtt{id}, v_0)^{-1}$$

of the Camassa-Holm shallow water equation (4.6)-(4.7) as well as for its parametrized version (4.3)-(4.4).

Proof of Theorem 1.1. The case $k = 1$ follows from Lemma 4.1 and the existence of solutions of the ordinary differential equations (2.7)-(2.8) and (2.9)-(2.10) (cf. Remark 4.3 below). The case $k = 0$ follows from the existence, uniqueness and dependence on parameters theorems for the KdV equation (cf. e.g. [3]) and the Burgers equation (cf. e.g. [4, 15]). The case $k \geq 2$ follows from Proposition 3.2 proved in Section 3. □

Proof of Theorem 1.3. As the statements for $k = 0$ and $k = 1$ are proved similarly we concentrate on $k = 1$ only. Taking $u_0 = c =$ *constant* one obtains from (4.3)-(4.4) that $u(t, x; c, a_0) \equiv c$. Solving (2.7)-(2.8) and (2.9)-(2.10) we then find that $\phi(t, x; c, a_0) = x + ct$ and $\alpha(t; c, a_0) = a_0 t$. Hence,

$$\exp_1((c, a_0)) = (\tau_c, a_0) \in \mathtt{Vir}$$

where τ_c denotes the translation $x \mapsto x + c$ on $\mathbb{T}$.

Our aim is to compute the Fréchet differential of the exponential map $(D\exp_1)|_{(c,a_0)} : \mathtt{vir} \to T_{(\tau_c,a_0)}\mathtt{Vir}$. To this end denote $\xi := D_2 u(t; u_0, a_0)(w)$ the partial directional derivative of the solution $u(t; u_0, a_0)$ with respect to the second variable u_0 in the direction $w \in C^\infty(\mathbb{T})$ at the point $(t; u_0, a_0)$ (cf. Appendix A), i.e. $\xi = \lim_{s\to 0}(u(t; u_0 + sw, a_0) - u(t; u_0, a_0))/s$. Since the partial derivative $D_2 u(t; u_0, a_0)$ is the restriction of the directional derivative of $u(t, \cdot, a_0) : H^l(\mathbb{T}) \to H^l(\mathbb{T})$ to $C^\infty(\mathbb{T})$ with $l \geq 3$ we compute ξ by working in the Hilbert space $H^l(\mathbb{T})$. As $u(t; u_0 + sw, a_0)$ satisfies (4.3) with initial datum $u|_{t=0} = u_0 + sw$ one obtains from Remark 4.2 and a differentiation with respect to s that $\xi(t, x)$ satisfies the linear PDE

$$\begin{aligned} A_1 \xi_t &= -2u_x A_1 \xi - 2\xi_x A_1 u - u A_1 \xi_x - \xi A_1 u_x + a_0 \xi_{xxx} && (4.11) \\ \xi|_{t=0} &= w && (4.12) \end{aligned}$$

[11] Let $1 \leq j \leq n$. We denote by $D_j f(x_1, ... x_n)$ the partial derivative of f with respect to the j-th variable at the point $(x_1, ..., x_n)$.

Taking $u_0 = c$ and using that $u(t,x;c,a_0) \equiv c$ we obtain from (4.11)-(4.12) that $\xi = \xi(t,x;c,a_0,w)$ satisfies the linear PDE

$$\begin{aligned} A_1\xi_t &= -2c\xi_x - cA_1\xi_x + a_0\xi_{xxx} && (4.13)\\ \xi|_{t=0} &= w && (4.14)\end{aligned}$$

If $a_0 = 2c$ the equation above becomes $A_1(\xi_t + 3c\xi_x) = 0$. As the operator $A_1 = 1 - \partial_x^2 : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ is a continuous bijection we obtain that $\xi_t + 3c\xi_x = 0$ and $\xi|_{t=0} = w$. Solving the latter equation one gets $\xi(t,x) = w(x - 3ct)$, i.e.

$$D_2u(t;c,2c)(w) = w(x - 3ct). \tag{4.15}$$

Our next goal is to compute the directional derivatives $D_2\phi(t;c,a_0)(w)$ and $D_2\alpha(t;c,a_0)(w)$. Proceeding as above we linearize the equations (2.7)-(2.8) and (2.9)-(2.10) at $u_0 = c$ and then find the directional derivatives by solving the corresponding linear equations.

Remark 4.3. *Note that for any $l \geq 1$ equation (2.7)-(2.8) can be regarded as a dynamical system (ODE) in the Hilbert space $H^l(\mathbb{T})$ depending on a parameter u from the Banach space $X := C^1([-T,T], H^{l+1}(\mathbb{T}))$*

$$\begin{aligned} \dot\phi &= F(t,\phi;u) && (4.16)\\ \phi|_{t=0} &= \mathtt{id} && (4.17)\end{aligned}$$

where $F(t,\phi;u) := u(t)\circ\phi$. Since the composition $H^{l+1}(\mathbb{T})\times H^l(\mathbb{T}) \to H^l(\mathbb{T})$, $(u,\phi) \to u\circ\phi$ is C^1, it follows that $F \in C^1([-T,T]\times H^l(\mathbb{T})\times X, H^l(\mathbb{T}))$ and hence (4.16)-(4.17) has a (unique) solution $\phi(t;u)$ in $H^l(\mathbb{T})$ that belongs to the space $C^1([-T,T]\times X, H^l(\mathbb{T}))$ (cf. [10, §3]).

Let $u(t;u_0,a_0)$ be the solution of (4.3)-(4.4). It follows from Lemma 4.1 and Remark 4.3 that the directional derivative $\eta := D_2\phi(t;u_0,a_0)(w)$ satisfies the variational equation

$$\begin{aligned} \eta_t &= u_x\circ\phi\cdot\eta + D_2u(t;u_0,a_0)(w)\circ\phi && (4.18)\\ \eta|_{t=0} &= 0 && (4.19)\end{aligned}$$

where $\phi = \phi(t;u_0,a_0)$ is the solution of (2.7)-(2.8) with $u = u(t;u_0,a_0)$. Taking $u_0 = c$ and $a_0 = 2c$ we have $u \equiv c$ and $\phi(t;x) = x + ct$ and obtain from (4.15) that $\eta = \eta|_{u_0=c,a_0=2c}$ satisfies $\eta_t = w((x+ct)-3ct) = w(x-2ct)$. Hence,

$$\eta = D_2\phi(t;c,2c)(w) = \int_0^t w(x - 2c\tau)\,d\tau. \tag{4.20}$$

As $\alpha(t; u_0, a_0) = a_0 t - \frac{1}{2}\int_0^t \int_0^1 u_x(\tau, \phi(\tau, x))\ d\log\phi_x(\tau, x)\ d\tau$ one concludes from $u_x = 0$ *and* $\phi_x = 1$ that

$$D_2\alpha(t; c, a_0) = 0\ . \tag{4.21}$$

Taking c sufficiently small so that $(u_0, a_0) = (c, 2c)$ is in the domain of definition of $\exp_1$ we obtain from (4.15), (4.20) and (4.21) that

$$(D\exp_1)|_{(c,2c)}(w, 0) = \Big(\int_0^1 w(x - 2c\tau)\ d\tau, 0\Big). \tag{4.22}$$

Finally, taking $c = c_n := \frac{1}{n}$, $w(x) = w_n(x) := \sin n\pi x$ and $N > 0$ sufficiently large one obtains from the formula above that for any $n \geq N$

$$(D\exp_1)|_{(c_n,2c_n)}(w_n, 0) = \Big(\int_0^1 w_n(x - 2c_n\tau)\ d\tau, 0\Big) = 0. \tag{4.23}$$

As $(c_n, 2c_n) = (1/n, 2/n) \to (0, 0)$ for $n \to 0$ one then obtains from Remark 5.4 that there is *no* neighborhood U of zero in `vir` so that $\exp_1$ is a C^1_F-diffeomorphism from U onto a neighborhood of the unital element e in `Vir`. $\square$

5 Appendix A: Calculus on Fréchet spaces

In this appendix we collect some definitions and notions from the calculus in Fréchet spaces. For more details we refer the reader to [12].

Fréchet spaces: Consider the pair $(X, \{||\cdot||_n\}_{n\in\mathbb{Z}_\geq})$ where X is a real vector space and $\{||\cdot||_n\}_{n\in\mathbb{Z}_\geq}$ is a countable collection of seminorms. We define a topology on X in the usual way using the collection of seminorms as follows: A basis of open neighborhoods of $0 \in X$ is given by the sets

$$U_{\epsilon,k_1,\ldots,k_s} := \{x \in X \mid ||x||_{k_j} < \epsilon\ \forall 1 \leq j \leq s\}$$

where $s, k_1, \ldots k_s \in \mathbb{Z}_\geq$ and $\epsilon > 0$. Then the topology on X is defined as the set of open sets generated by the sets $x + U_{\epsilon,k_1,\ldots,k_s}$ with $x \in X$, $s, k_1, \ldots, k_s \in \mathbb{Z}_\geq$ and $\epsilon > 0$ arbitrary. In this way X becomes a topological vector space. Note that a sequence x_k converges to x in X iff for any $n \geq 0$, $||x_k - x||_n \to 0$ as $k \to \infty$.

Let X be a topological vector space whose topology is induced from the countable system of seminorms $\{||\cdot||_n\}_{n\in\mathbb{Z}_\geq}$. Then X is *Hausdorff* iff for any $x \in X, ||x||_n = 0\ \forall n \in \mathbb{Z}_\geq$ implies $x = 0$. A sequence $(x_k)_{k\in\mathbb{N}}$ is called *Cauchy* iff it is a Cauchy sequence with respect to any of the seminorms $||\cdot||_n$, $n \in \mathbb{Z}_\geq$. By definition, X is complete iff every Cauchy sequence converges in X.

Definition 5.1. *A pair $(X, \{||\cdot||_n\}_{n\in\mathbb{Z}_\geq})$ consisting of a topological vector space X and a countable system of seminorms $\{||\cdot||_n\}_{n\in\mathbb{Z}_\geq}$ is called a* Fréchet space [12] *iff the topology of X is the one induced by $\{||\cdot||_n\}_{n\in\mathbb{Z}_\geq}$ and X is Hausdorff and complete.*

C^1_F*-differentiability:* Let $f : U \subseteq X \to Y$ be a map from an open set U of a Fréchet space X to a Fréchet space Y.

Definition 5.2. *The* (directional) derivative *of f at the point $x \in U$ in the direction $h \in X$ is*

$$D_x f(h) := \lim_{\epsilon\to 0}(f(x + \epsilon h) - f(x))/\epsilon \in Y \tag{5.1}$$

where the limit is taken with respect to the Fréchet topology of Y.

If the directional derivative $D_x f(h)$ exists then we say that f *is differentiable at x in the direction h.*

Definition 5.3. *If the directional derivative $D_x f(h)$ exists for any $x \in U$ and any $h \in X$ and the map*

$$(x, h) \mapsto D_x f(h),\ U \times X \to Y$$

is continuous with respect to the Fréchet topology on $U \times X$ and Y then f is called continuously differentiable on U *or* C^1_F-smooth. *The space of all such maps is denoted by $C^1_F(U, Y)$.*[13] *A map $f : U \to V$ from an open set $U \subseteq X$ onto an open set $V \subseteq Y$ is called* a C^1_F-diffeomorphism *if f is a homeomorphism and f as well as f^{-1} are C^1_F-smooth.*

[12] Unlike for the standard notion of a Fréchet space, in this definition the countable system of seminorms defining the topology of X is a part of the structure of the space.

[13] Note that even in the case where X and Y are Banach spaces this definition of continuous differentiability is weaker than the usual one (cf. [12]). In order to distinguish it from the classical one we write C^1_F instead of C^1. We refer to [12] for a discussion of the reasons to introduce the notion of C^1_F-differentiability.

Remark 5.4. *Using the chain rule one easily obtains that for any $x \in U$ the directional derivative $D_x f : X \to Y$ of a C^1_F-diffeomorphism $f : U \to V$ is a linear isomorphism.*

We refer to [12] for the definitions of the higher derivatives ($k \geq 2$)

$$D^k_\bullet f : U \times \underbrace{X \times ... \times X}_{k} \to Y, \ (x, h_1, ..., h_k) \mapsto D^k_x f(h_1, ..., h_k)$$

and the definition of the space $C^k_F(U, Y)$. We only remark that as in the classical calculus in Banach spaces, $f \in C^k_F(U, Y)$ implies that the k'th derivative $D^k_x f : \underbrace{X \times ... \times X}_{k} \to Y$ is a symmetric, k-linear map for any $x \in U$.

In this paper we consider mainly the following spaces:

Fréchet space $C^\infty(\mathbb{T})$. The space $C^\infty(\mathbb{T}) \equiv C^\infty(\mathbb{T}, \mathbb{R})$ denotes the real vector space of real-valued C^∞-smooth, 1-periodic functions $u : \mathbb{R} \to \mathbb{R}$. The topology on $C^\infty(\mathbb{T})$ is induced by the countable system of Sobolev norms: $||u||_n := \Big(\sum_{j=0}^n \int_0^1 u^{(j)}(x)^2 \, dx \Big)^{1/2}$ with $n \geq 0$.

Fréchet manifold $\mathcal{D}(\mathbb{T})$. By definition, $\mathcal{D}(\mathbb{T})$ denotes the group of C^∞-smooth positively oriented diffeomorphisms of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. A Fréchet manifold structure on $\mathcal{D}(\mathbb{T})$ can be introduced as follows: Passing in domain and target to the universal cover $\mathbb{R} \to \mathbb{T}$, any element ϕ of $\mathcal{D}(\mathbb{T})$ gives rise to a smooth diffeomorphism of $\mathbb{R}$ in $C^\infty(\mathbb{R}, \mathbb{R})$, again denoted by ϕ, satisfying the *normalization condition*

$$-1/2 < \phi(0) < 1/2 \tag{5.2}$$

or

$$0 < \phi(0) < 1. \tag{5.3}$$

The function $f(x) := \phi(x) - x$ is 1-periodic and therefore lies in $C^\infty(\mathbb{T})$. Moreover $f'(x) > -1$ for any $x \in \mathbb{R}$. The normalizations (5.2) and (5.3) give rise to two charts U_1, U_2 of $\mathcal{D}(\mathbb{T})$ with $U_1 \cup U_2 = \mathcal{D}(\mathbb{T})$

$$J_j : V_j \to U_j, f \mapsto \phi := \mathtt{id} + f$$

where

$$\begin{aligned} V_1 &:= \{f \in C^\infty(\mathbb{T}) \mid -1/2 < f(0) < 1/2 \text{ and } f' > -1\} \subseteq C^\infty(\mathbb{T}) \\ V_2 &:= \{f \in C^\infty(\mathbb{T}) \mid 0 < f(0) < 1 \text{ and } f' > -1\} \subseteq C^\infty(\mathbb{T}). \end{aligned}$$

As V_1, V_2 are both open sets in the Fréchet space $C^\infty(\mathbb{T})$, the construction above gives an atlas of Fréchet charts of $\mathcal{D}(\mathbb{T})$. In this way, $\mathcal{D}(\mathbb{T})$ is a Fréchet manifold modeled on $C^\infty(\mathbb{T})$.

Hilbert manifold $\mathcal{D}^s(\mathbb{T})$ *($s \geq 2$).* $\mathcal{D}^s(\mathbb{T})$ denotes the group of positively oriented bijective transformations of $\mathbb{T}$ of class H^s. By definition, a bijective transformation ϕ of $\mathbb{T}$ is of class H^s iff the lift $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ of ϕ, determined by the normalization, $0 \leq \tilde{\phi}(0) < 1$, and its inverse $\tilde{\phi}^{-1}$ both lie in $H^s_{loc}(\mathbb{R}, \mathbb{R})$. As for $\mathcal{D}(\mathbb{T})$ one can introduce an atlas for $\mathcal{D}^s(\mathbb{T})$ with two charts in $H^s(\mathbb{T})$, making $\mathcal{D}^s(\mathbb{T})$ a Hilbert manifold modeled on $H^s(\mathbb{T})$.

Hilbert approximations: Assume that for a given Fréchet space X there is a sequence of Hilbert spaces $\{(X_n, ||\cdot||_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ such that

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq ... \supseteq X \quad \text{and} \quad X = \cap_{n=0}^{\infty} X_n$$

where $\{||\cdot||_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is a sequence of norms inducing the topology on X so that $||x||_0 \leq ||x||_1 \leq ||x||_2 \leq ... \; \forall x \in X$. Such a sequence of Hilbert spaces $\{(X_n, ||\cdot||_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ is called a *Hilbert approximation* of the Fréchet space X. For Fréchet spaces admitting Hilbert approximations one can prove the following version of the inverse function theorem.

Proposition 5.5. *Let X and Y be Fréchet spaces admitting the Hilbert approximations $\{(X_n, ||\cdot||_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ and $\{(Y_n, |\cdot|_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ respectively. Assume that $f : U_0 \to V_0$ is a C^1-diffeomorphism between the open sets $U_0 \subseteq X_0$ and $V_0 \subseteq Y_0$ of the Hilbert spaces X_0 and Y_0 respectively. Define the sets $U_n := U_0 \cap X_n$ and $V_n := V_0 \cap Y_n$ and assume that the following properties are satisfied for any $n \geq 0$:*

(a) if $x \in U_0$ then $f(x) \in V_n$ iff $x \in X_n$;

(b) the restriction $f|_{U_n} : U_n \to Y_n$ is C^1-smooth;

(c) for any $x \in U_n$, $d_x f(X_n \setminus X_{n+1}) \subseteq Y_n \setminus Y_{n+1}$.

Then $U := U_0 \cap X$ and $V := V_0 \cap Y$ are open sets in X and Y respectively with $f(U) \subseteq V$ and the mapping $f_\infty := f|_U : U \to V$ is a C^1_F-diffeomorphism.

Remark 5.6. *The same results hold for approximations of X and Y by Banach spaces instead of Hilbert spaces.*

Proof of Proposition 5.5. Note first that for any $n \geq 0$, the set U_n is open in X_n and the set V_n is open in Y_n. As $f : U_0 \to V_0$ is bijective, (a) implies that $f|_{U_n} : U_n \to V_n$ is well defined and bijective as well. Indeed, the injectivity of $f|_{U_n}$ follows from the injectivity of f. As f is bijective, for any $y \in V_n \subset V_0$ there exists a unique element $x \in U_0$ such that $f(x) = y \in V_n$. Then according to (a), $x \in X_n$, hence $x \in U_n = U_0 \cap X_n$. Thus for any $n \geq 0$, $f|_{U_n} : U_n \to V_n$, and therefore $f_\infty : U \to V$, are bijective.

For any $n \in \mathbb{Z}_{\geq 0}$, let $f_n := f|_{U_n}$. According to (b), $f_n : U_n \to V_n$ is then a C^1-smooth bijective map. In order to prove that $f_n^{-1} : V_n \to U_n$ is C^1-smooth as well we will use the inverse function theorem in Hilbert spaces. Take $x \in U_n$ and consider the differential $d_x f_n : X_n \to Y_n$. As

$$(d_x f_0)|_{X_n} = d_x f_n$$

and $f_0 : U_0 \to V_0$ is a C^1-diffeomorphism one concludes that $d_x f_n$ is injective. We prove by induction that $d_x f_n : X_n \to Y_n$ is onto. For $k = 0$ the statement is true by assumption. Assume that it holds for any $k \leq n-1$. As $d_x f_{n-1}$ is onto, for any $\eta \in Y_n \subset Y_{n-1}$ there exists $\xi \in X_{n-1}$ such that $d_x f_n(\xi) = \eta \in Y_n$. Then (c) implies that $\xi \in X_n$. Hence, $d_x f_n : X_n \to Y_n$ is bijective. By the inverse function theorem for Hilbert spaces, $f_n : U_n \to Y_n$ is a C^1-diffeomorphism. In particular, for any $n \geq 0$ the maps

$$U_n \times X_n \to Y_n, \ (x, \xi) \mapsto d_x f_n(\xi) \tag{5.4}$$

and

$$V_n \times Y_n \to X_n, \ (y, \eta) \mapsto d_y(f_n^{-1})(\eta) \tag{5.5}$$

are continuous. As for any $x \in U$ and $n \geq 0$,

$$D_x f_\infty = (d_x f_n)|_X$$

one gets from (5.4)-(5.5) that

$$U \times X \to Y, \ (x, \xi) \mapsto D_x f_\infty(\xi) \quad \text{and} \quad V \times Y \to X, \ (y, \eta) \mapsto D_y(f_\infty^{-1})(\eta)$$

are continuous. In particular one concludes that

$$f_\infty : U \to V$$

is a C_F^1-diffeomorphism. □

6 Appendix B: Euler equation on $\mathtt{vir}$

In this appendix we derive the Euler-Lagrange equations of geodesics of the right-invariant weak Riemannian metrics $\mu^{(k)}$ (cf. (2.4)) on the Virasoro group $\mathtt{Vir}$ given by the action principle. The cases $k = 0, 1$ were considered in [22, 16] in a somewhat formal way using a purely algebraic approach (cf. [1]). At the end of the appendix we derive for any $k \geq 0$ a conservation law for the geodesic flow of the metric $\mu^{(k)}$.

Let $\gamma(s,t) = (\phi(s,t), \alpha(s,t)) \in \mathtt{Vir}$ be a C^2_F-smooth variation ($-\epsilon < s < \epsilon, 0 \leq t \leq T$)

$$\gamma : (-\epsilon, \epsilon) \times [0,T] \to \mathtt{Vir} \tag{6.1}$$

of the C^2_F-smooth curve $\gamma(t) \equiv \gamma(0,t) : [0,T] \to \mathtt{Vir}$ such that for any $-\epsilon < s < \epsilon$

$$\gamma(s,0) \equiv e \quad \text{and} \quad \gamma(s,T) \equiv \gamma(T) \tag{6.2}$$

where e denotes the unital element in $\mathtt{Vir}$, $e = (\mathtt{id}, 0)$. It follows from the multiplication (2.1) on the Virasoro group $\mathtt{Vir}$ that the derivative $d_e R_{(\phi,\alpha)}$ of the right-translation $R_{(\phi,\alpha)} : \mathtt{Vir} \to \mathtt{Vir}$, $(\psi, \beta) \mapsto (\psi,\beta)\circ(\phi,\alpha)$ at $(\psi,\beta) = e$, $d_e R_{(\phi,\alpha)} : T_e\mathtt{Vir} \to T_{(\phi,\alpha)}\mathtt{Vir}$, is given by

$$d_e R_{(\phi,\alpha)}(u,a) = \Big(u \circ \phi, a - \frac{1}{2}\int_0^1 u_x(\phi(x))\, d\log\phi_x(x)\Big)$$

where $(\phi,\alpha) \in \mathtt{Vir}$ and $(u,a) \in \mathtt{vir}(\cong T_e\mathtt{Vir})$. In particular,

$$\begin{aligned}&(d_e R_{\gamma(s,t)})^{-1}(\dot\gamma(s,t)) = \\ &= \Big(\phi_t(s,t)\circ\phi(s,t)^{-1}, \dot\alpha(s,t) + \tfrac{1}{2}\int_0^1 \tfrac{\phi_{tx}(s,t)}{\phi_x(s,t)} d\log\phi_x(s,t)\Big)\end{aligned}$$

where $\dot\gamma(s,t) = \frac{\partial\gamma(s,t)}{\partial t}$, $\partial_t \equiv \partial/\partial t$ and $\partial_x \equiv \partial/\partial x$. Hence, for $k \geq 0$

$$\begin{aligned}\mu^{(k)}_{\gamma(s,t)}(\dot\gamma(s,t),\dot\gamma(s,t)) &= \sum_{j=0}^k \int_0^1 (\partial_x^j(\phi_t(s,t)\circ\phi(s,t)^{-1}))^2\, dx \\ &+ \Big(\dot\alpha(s,t) + \frac{1}{2}\int_0^1 \frac{\phi_{tx}(s,t)}{\phi_x(s,t)}\, d\log\phi_x(s,t)\Big)^2\end{aligned}$$

and the *action functional* is

$$E_{\mu^{(k)}}(\gamma(s,\cdot)) := \sum_{j=0}^k E_j(\gamma(s,\cdot)) + A(\gamma(s,\cdot)) \tag{6.3}$$

where for $0 \leq j \leq k$

$$E_j(\gamma(s,\cdot)) := \frac{1}{2}\int_0^T \int_0^1 (\partial_x^j(\phi_t(s,t)\circ\phi(s,t)^{-1}))^2 \, dxdt$$

and

$$A(\gamma(s,\cdot)) := \frac{1}{2}\int_0^T \left(\dot\alpha(s,t) + \frac{1}{2}\int_0^1 \frac{\phi_{tx}(s,t)}{\phi_x(s,t)}\, d\log\phi_x(s,t)\right)^2 dt.$$

Denoting $u(t) := \phi_t(t)\circ\phi^{-1}(t)$ where $\phi(t) = \phi(0,t)$ we obtain

$$\phi_t(t) = u(t)\circ\phi(t). \tag{6.4}$$

Introduce the variations $\delta E_j(\gamma) := \frac{d}{ds}|_{s=0} E_j(\gamma(s,\cdot))$, $\delta\phi := \frac{d}{ds}|_{s=0}\phi(s,t)$, and $\delta\alpha := \frac{d}{ds}|_{s=0}\alpha(s,t)$.

Remark 6.1. *As $\gamma \in C_F^2((-\epsilon,\epsilon)\times[-T,T],\mathtt{Vir})$ it follows from the definition of the space $C_F^2((-\epsilon,\epsilon)\times[-T,T],\mathtt{Vir})$ that $\alpha = \alpha(s,t)$ lies in $C^2((-\epsilon,\epsilon)\times[-T,T])$ and $\phi(s,t) = \phi(s,t,x)$ considered as a $\mathbb{R}$-valued function of s, t and x is continuous and for any $j \geq 0$ the partial derivatives $\partial_s\partial_x^j\phi(s,t,x)$, $\partial_t\partial_x^j\phi(s,t,x)$, $\partial_s^2\partial_x^j\phi(s,t,x)$, $\partial_t^2\partial_x^j\phi(s,t,x)$ and $\partial_s\partial_t\partial_x^j\phi(s,t,x)$ are continuous. Moreover in the expressions above all the partial derivative operators ∂_s, ∂_t, and ∂_x commute.*

Using (6.4), $\phi(s,t)\circ\phi^{-1}(s,t) = \mathtt{id}$ and the change of variables $y = \phi(t)^{-1}(x)$ one obtains

$$\begin{aligned}
\delta E_j(\gamma) &= \int_0^T\int_0^1 \partial_x^j u\cdot\partial_x^j\Big((\delta\phi)_t\circ\phi^{-1} - \phi_{tx}\circ\phi^{-1}\cdot\delta\phi^{-1}\Big)\, dxdt \\
&= (-1)^j\int_0^T\int_0^1 \partial_x^{2j}u\cdot\left((\delta\phi)_t\circ\phi^{-1} - \frac{\phi_{tx}\circ\phi^{-1}}{\phi_x\circ\phi^{-1}}\cdot(\delta\phi)\circ\phi^{-1}\right) dxdt \\
&= (-1)^j\int_0^T\int_0^1 (\partial_x^{2j}u)\circ\phi\cdot\phi_x\cdot(\delta\phi)_t\, dxdt \\
&+ (-1)^{j+1}\int_0^T\int_0^1 (\partial_x^{2j}u)\circ\phi\cdot\phi_{tx}\cdot\delta\phi\, dxdt \\
&= (-1)^{j+1}\int_0^T\int_0^1 \frac{\delta E_j}{\delta\phi}\cdot\delta\phi\, dxdt \qquad (6.5)
\end{aligned}$$

where

$$
\begin{aligned}
\frac{\delta E_j}{\delta\phi} &:= ((\partial_x^{2j}u)\circ\phi\cdot\phi_x)_t + (\partial_x^{2j}u)\circ\phi\cdot\phi_{tx} \\
&= (\partial_x^{2j}u_t)\circ\phi\cdot\phi_x + 2(\partial_x^{2j}u)\circ\phi\cdot\phi_{tx} \\
&+ (\partial_x^{2j}u_x)\circ\phi\cdot\phi_t\cdot\phi_x \\
&= \Big((\phi_x\circ\phi^{-1})\cdot(\partial_x^{2j}u_t + 2u_x\partial_x^{2j}u + u\partial_x^{2j}u_x)\Big)\circ\phi\ . \qquad (6.6)
\end{aligned}
$$

Here we have used that $\phi_t = u\circ\phi$ and thus $u_x\circ\phi = \frac{\phi_{tx}}{\phi_x}$. Analogously, one has

$$
\begin{aligned}
\delta A(\gamma) &= \int_0^T \Big(\dot\alpha(t) + \frac{1}{2}\int_0^1 \frac{\phi_{tx}}{\phi_x}\, d\log\phi_x\Big)\cdot\Big((\delta\alpha)_t + \frac{1}{2}\delta\int_0^1 \frac{\phi_{tx}}{\phi_x}\, d\log\phi_x\Big)dt \\
&= -\int_0^T \Big(\dot\alpha(t) + \frac{1}{2}\int_0^1 \frac{\phi_{tx}}{\phi_x}\, d\log\phi_x\Big)_t\cdot\delta\alpha\; dt \qquad (6.7) \\
&+ \frac{1}{2}\int_0^T \Big(\dot\alpha(t) + \frac{1}{2}\int_0^1 \frac{\phi_{tx}}{\phi_x}\, d\log\phi_x\Big)\Big(\delta\int_0^1 \frac{\phi_{tx}}{\phi_x}\, d\log\phi_x\Big)\, dt. \qquad (6.8)
\end{aligned}
$$

It follows from (6.3), (6.5), and (6.7) that for $\delta\phi = 0$ and $\delta\alpha$ arbitrary

$$
\dot\alpha(t) + \frac{1}{2}\int_0^1 \frac{\phi_{tx}}{\phi_x}\, d\log\phi_x = a = const \qquad (6.9)
$$

where $a = \dot\alpha(0)$. In particular,

$$
\dot\alpha(t) = a - \frac{1}{2}\int_0^1 \frac{\phi_{tx}}{\phi_x}\, d\log\phi_x\ . \qquad (6.10)
$$

Provided (6.10) is satisfied

$$
\begin{aligned}
\delta A(\gamma) &= \frac{a}{2}\delta\int_0^T\int_0^1 \frac{\phi_{tx}}{\phi_x}\, d\log\phi_x\; dt \\
&= \frac{a}{2}\delta\int_0^T\int_0^1 (\log\phi_x)_t(\log\phi_x)_x\; dxdt \\
&= \frac{a}{2}\int_0^T\int_0^1 \Big(\frac{(\delta\phi)_x}{\phi_x}\Big)_t(\log\phi_x)_x\; dxdt \\
&+ \frac{a}{2}\int_0^T\int_0^1 (\log\phi_x)_t\Big(\frac{(\delta\phi)_x}{\phi_x}\Big)_x\; dxdt \\
&= -a\int_0^T\int_0^1 (\log\phi_x)_{tx}\Big(\frac{(\delta\phi)_x}{\phi_x}\Big)\; dxdt.
\end{aligned}
$$

As $u_x \circ \phi = \frac{\phi_{tx}}{\phi_x} = (\log \phi_x)_t$ we get that $u_{xx} \circ \phi \cdot \phi_x = (\log \phi_x)_{tx}$ and hence

$$\begin{aligned} \delta A(\gamma) &= -a \int_0^T \int_0^1 u_{xx} \circ \phi \cdot (\delta\phi)_x \, dxdt \\ &= a \int_0^T \int_0^1 u_{xxx} \circ \phi \cdot \phi_x \cdot \delta\phi \, dxdt. \end{aligned} \tag{6.11}$$

Finally, (6.3), (6.5) and (6.11) show that $\delta E(\gamma) = 0$ iff $\gamma : [-T, T] \to \mathtt{Vir}$ satisfies the equations:

$$\phi_t(t) = u(t) \circ \phi(t) \tag{6.12}$$

$$\dot\alpha(t) = a - \frac{1}{2} \int_0^1 \frac{\phi_{tx}}{\phi_x} \, d \log \phi_x \tag{6.13}$$

$$A_k u_t = -2u_x A_k u - u A_k u_x + a u_{xxx} \tag{6.14}$$

where $A_k := \sum_{j=0}^k (-1)^j \partial_x^{2j}$ and $a = \dot\alpha(0)$. The system (6.12)-(6.14) can be divided into two parts: the *Euler equation* part which is the equation for the curve $(u(t), a(t)) := (d_e R_{\gamma(t)})^{-1}(\dot\gamma(t))$ in the Lie algebra $\mathtt{vir}$,

$$\begin{aligned} A_k u_t &= -2u_x A_k u - u A_k u_x + a u_{xxx} \\ \dot a = 0 & \end{aligned} \tag{6.15}$$

and the translation part

$$\phi_t(t) = u(t) \circ \phi(t) \tag{6.16}$$

$$\dot\alpha(t) = a - \frac{1}{2} \int_0^1 \frac{\phi_{tx}}{\phi_x} \, d \log \phi_x \tag{6.17}$$

coming from the right-translation $d_e R_{\gamma(t)}(u(t), a(t)) = \dot\gamma(t)$. Hence we have derived the system of equations (2.7)-(2.12) stated in Section 2.

We end this section by deriving for any $k \geq 0$ a conservation law for the geodesic flow of the metric $\mu^{(k)}$ on Vir. Define $I_k : TVir \to C^\infty(\mathbb{T})$, given for any $(v, b) \in T_{(\psi,\beta)}\mathtt{Vir}$ by

$$I_k(v, b) := \psi_x^2 \cdot (A_k u) \circ \psi - a S(\psi)$$

where $(u, a) = (d_e R_{(\psi,\beta)})^{-1}(v, b) \in \mathtt{vir}$. In the following lemma, we verify that the function $I_k(v, b)$ is invariant for the geodesic flow of $\mu^{(k)}$.

Lemma 6.2. *If $\gamma : [-T, T] \to$ `Vir`, $\gamma(t) = (\phi(t), \alpha(t))$ is a geodesic of the right-invariant Riemannian metric $\mu^{(k)}$ on the Virasoro group `Vir` (cf. Definition 2.4) then*

$$I_k(\dot{\gamma}(t)) = \phi_x(t)^2 \cdot (A_k u(t)) \circ \phi(t) - aS(\phi(t)) \tag{6.18}$$

is independent of t where $u(t) := \phi_t(t) \circ \phi^{-1}(t)$, $a := \dot{\alpha}(0)$ and $S(\phi(t))$ denotes the Schwarzian derivative $(\phi_x(t)\phi_{xxx}(t) - 3\phi_{xx}^2(t)/2)/\phi_x^2(t)$.

Proof. Using equation (6.14) and the identity $\phi_t(t) = u(t) \circ \phi(t)$ one obtains

$$\begin{aligned} I_k(\dot{\gamma}(t))_t &= 2\phi_x \cdot \phi_{tx} \cdot (A_k u) \circ \phi + \phi_x^2 \cdot (A_k u_t) \circ \phi \\ &+ \phi_x^2 \cdot (A_k u_x) \circ \phi \cdot \phi_t - aS(\phi)_t \\ &= 2\phi_x^2 \cdot u_x \circ \phi \cdot (A_k u) \circ \phi + \phi_x^2 \cdot (-2u_x A_k u - u A_k u_x + a u_{xxx}) \circ \phi \\ &+ \phi_x^2 \cdot (A_k u_x) \circ \phi \cdot u \circ \phi - aS(\phi)_t \\ &= a(u_{xxx} \circ \phi \cdot \phi_x^2 - S(\phi)_t) \,. \end{aligned} \tag{6.19}$$

As $\phi_t = u \circ \phi$ one has $u_x \circ \phi = \frac{\phi_{tx}}{\phi_x} = (\log \phi_x)_t$. Hence,

$$u_{xx} \circ \phi \cdot \phi_x = \Big(\frac{\phi_{xx}}{\phi_x} \Big)_t \tag{6.20}$$

and

$$u_{xxx} \circ \phi \cdot \phi_x^2 + u_{xx} \circ \phi \cdot \phi_{xx} = \Big(\frac{\phi_{xx}}{\phi_x} \Big)_{xt} \,. \tag{6.21}$$

Finally, (6.21) and (6.20) give

$$\begin{aligned} u_{xxx} \circ \phi \cdot \phi_x^2 &= \Big(\frac{\phi_{xx}}{\phi_x} \Big)_{xt} - \frac{\phi_{xx}}{\phi_x} \Big(\frac{\phi_{xx}}{\phi_x} \Big)_t \\ &= \Big(\Big(\frac{\phi_{xx}}{\phi_x} \Big)_x - \frac{1}{2} \Big(\frac{\phi_{xx}}{\phi_x} \Big)^2 \Big)_t \\ &= S(\phi)_t \end{aligned} \tag{6.22}$$

which together with (6.19) implies that $(I_k(\gamma(t)))_t = 0$. $\square$

7 Appendix C: Lie group exponential map for $\mathtt{Vir}$

In this appendix we prove that the Lie group exponential map of $\mathtt{Vir}$,

$$\exp_{\mathtt{Lie}}^{\mathtt{Vir}} : \mathtt{vir} \to \mathtt{Vir}\,,$$

is *not* locally onto near the unital element e of $\mathtt{Vir}$, i.e. there are elements in $\mathtt{Vir}$ arbitrarily close to e which are not in the image of $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}$. The value of $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}$ at $(u,a) \in \mathtt{vir}$ is defined as the time 1-map of the flow $t \mapsto (\phi(t), \alpha(t))$ corresponding to the right invariant vector field induced by $(u,a) \in \mathtt{vir}$,

$$\begin{aligned}(\phi_t, \alpha_t) &= d_e R_{(\phi,\alpha)}(u,a)\\ &= \Big(u \circ \phi, a - \frac{1}{2}\int_0^1 u_x \circ \phi \; d \log \phi_x\Big) \end{aligned} \tag{7.1}$$

$$(\phi, \alpha)|_{t=0} = (\mathtt{id}, 0)\,. \tag{7.2}$$

Using the Hilbert approximation $(\mathtt{vir}_l)_{l \geq 1}$ of $\mathtt{vir}$ and the fact that $\mathcal{D}$ is the diffeomorphism group of the *compact* manifold $\mathbb{T}$ one concludes that there exists a unique solution of the above initial value problem and that it is defined globally in time. Hence $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}$ is well defined and it turns out to be C_F^∞-smooth. Kopell [17] (see also [12] or [21]) proved that the Lie group exponential map $\exp_{\mathtt{Lie}}^{\mathcal{D}}$ of the diffeomorphism group is not locally onto near the unital element of $\mathcal{D}$. This result can be used to prove a similar result for $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}$.

Proposition 7.1 *The map* $\exp_{\mathtt{Lie}}^{\mathtt{Vir}} : \mathtt{vir} \to \mathtt{Vir}$ *is not locally onto near the unital element of* $\mathtt{Vir}$*, i.e. there are elements arbitrarily close to* e *which are not in the image of* $\exp_{\mathtt{Lie}}^{\mathtt{Vir}}$.

Proof. The Lie group exponential map $\exp_{\mathtt{Lie}}^{\mathcal{D}} : T_{\mathtt{id}}\mathcal{D} \to \mathcal{D}$ for the diffeomorphisms group $\mathcal{D}$ is defined to be the time 1-map of the flow given by $(u \in T_{\mathtt{id}}\mathcal{D} \cong C^\infty(\mathbb{T}))$

$$\begin{aligned}\phi_t &= d_{\mathtt{id}} R_\phi u = u \circ \phi\\ \phi|_{t=0} &= \mathtt{id}\end{aligned}$$

where here, $R_\phi : \mathcal{D} \to \mathcal{D}$ denotes the right translation on $\mathcal{D}$. It then follows that

$$\exp_{\mathtt{Lie}}^{\mathcal{D}} = \pi \circ \exp_{\mathtt{Lie}}^{\mathtt{Vir}} |_{T_{\mathtt{id}}\mathcal{D} \times \{0\}} \tag{7.3}$$

and

$$\pi \circ \exp_{\mathtt{Lie}}^{\mathtt{Vir}}(\mathtt{vir}) = \exp_{\mathtt{Lie}}^{\mathcal{D}}(T_{\mathtt{id}}\mathcal{D}) \tag{7.4}$$

where $\pi : \mathtt{Vir} = \mathcal{D} \times \mathbb{R} \to \mathcal{D}$ is the projection onto the first component. By [17] (see also [12, p. 123] or [21, p. 1018]) $\exp_{\mathtt{Lie}}^{\mathcal{D}}$ is *not* locally onto near the unital element of $\mathcal{D}$. We then conclude from (7.4) that $\exp_{Lie}^{\mathtt{Vir}}$ is not locally onto near the unital element of $\mathtt{Vir}$ as well. $\square$

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